GRAVITATIONAL FIELDS

AND SIMPLE HARMONIC MOTION

The Problems

See these pages for the problems.

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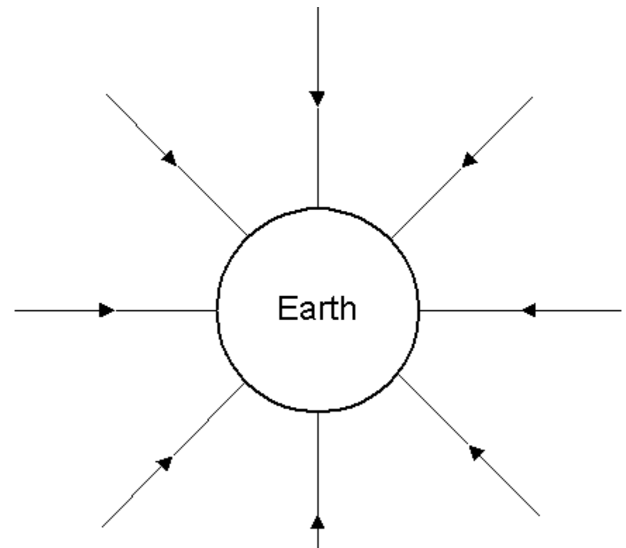
Gravitational fields

The concept of *field* is a key idea in physics. Let's apply it to gravitation.

The diagram illustrates a cross-section of the gravitational field of the Earth or any spherical celestial object. *Lines of force* are used to represent the field. Here's how the lines are drawn and what they tell us.



Gravitational field of the Earth

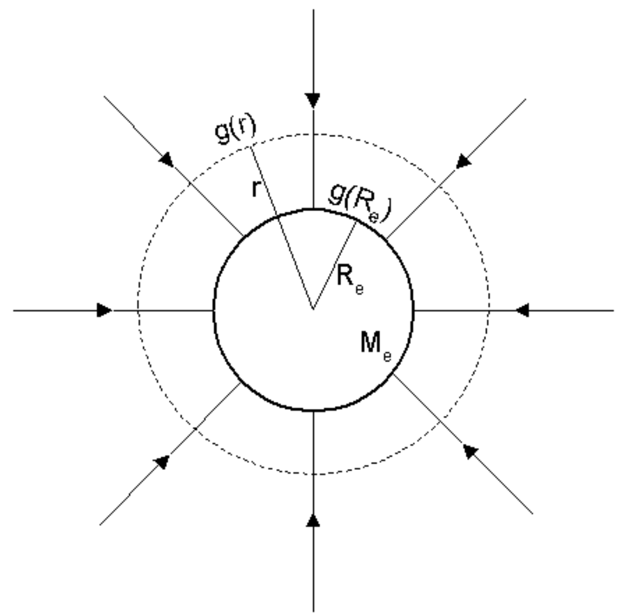
- At a given distance from the Earth, the lines are uniformly spaced. This indicates uniform field intensity, g. The value of g is the same at equal distances from the center of Earth.
- The spacing of the field lines increases with distance from the Earth. This indicates that the field intensity weakens as distance increases.
- The direction of the field (indicated by the arrows) is the direction of the force on a small mass placed in the field.

Gravitational fields, p. 2

A note about the symbol, g: Up to now, we've taken g to equal 9.8 N/kg at the surface of the Earth. This is only a special instance. More generally, g is a function of the distance, r, from the center of the Earth: $g(r) = GM_e/r^2$, where M_e is the mass of the Earth.

Note what $g(r)$ depends on:

- Distance from the center of the Earth, r. In the diagram, note that $g(r)$ is the same at all points on the dashed line equidistant from the center of the Earth. However $g(r) < g(R_e)$, since $r > R_e$.
- Mass of the Earth: The Earth is the source of the field. The gravitational field intensity depends on the mass of the source object.



Gravitational fields, p. 3

Next we relate the concept of force to that of field. In order for there to be a force, there must be an object in the field. Suppose we place a small mass, m, at distance, r. The gravitational force on the mass is $F_g(r)$.

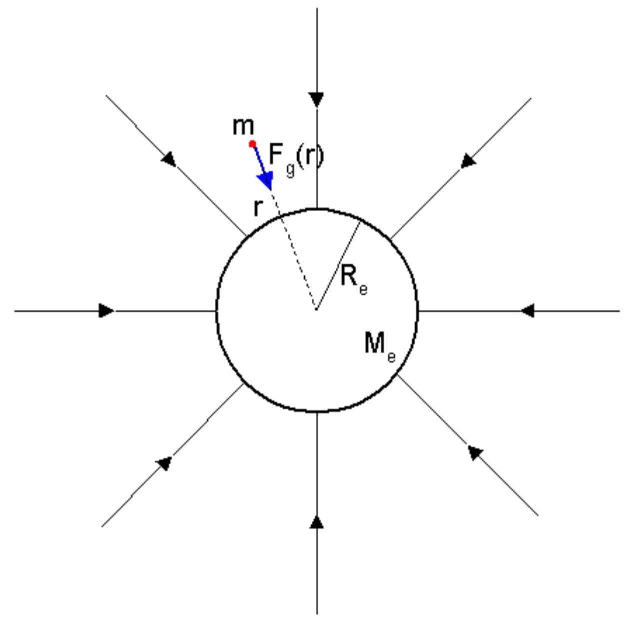
From the gravitational force law, we know that

$$F_g(r) = GM_e m/r^2.$$

Substituting $g(r) = GM_e/r^2$, we have

$$F_g(r) = mg(r).$$

The gravitational force on mass m at distance r from the center of the Earth is therefore the product of m and the gravitational field intensity at distance, r. This is how one calculates gravitational force given the values of m and $g(r)$.



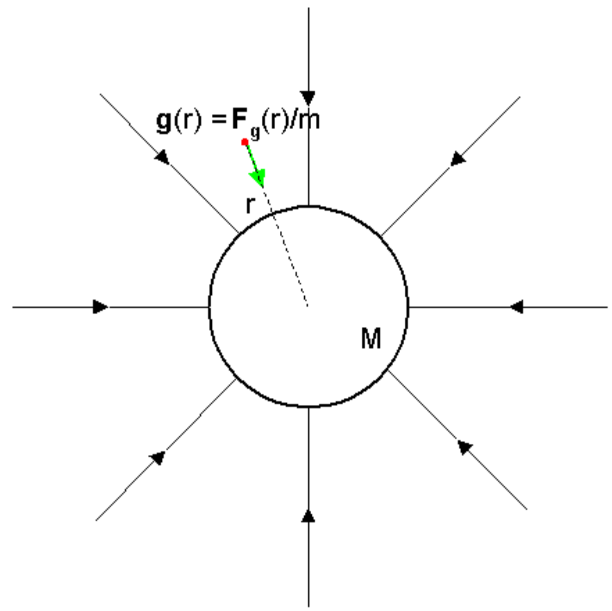
Gravitational fields, p. 4

The formula, $g(r) = GM/r^2$, is a special case of gravitational field for a spherically symmetric source object of mass, *M*. The field concept is used more generally to describe the fields of mass distributions. Here's the definition.

The gravitational field, $\boldsymbol{g}$, is the force per unit mass, $\boldsymbol{F_g}/m$, that would be exerted on a small mass, *m*, placed in the field.

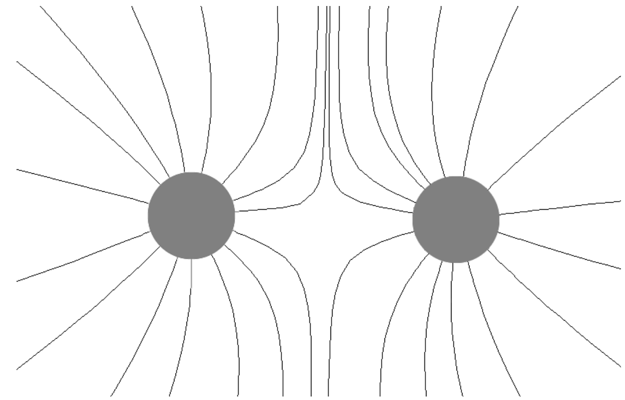
Symbolically, $\boldsymbol{g} = \boldsymbol{F_g}/m$. Note these things about the definition:

- We denote $\boldsymbol{g}$ and $\boldsymbol{F_g}$ in boldface to indicate that they're vectors. The direction of $\boldsymbol{g}$ is the same as the direction of the force.
- The mass, *m*, must be small enough that it has a negligible effect on the field of the source object.

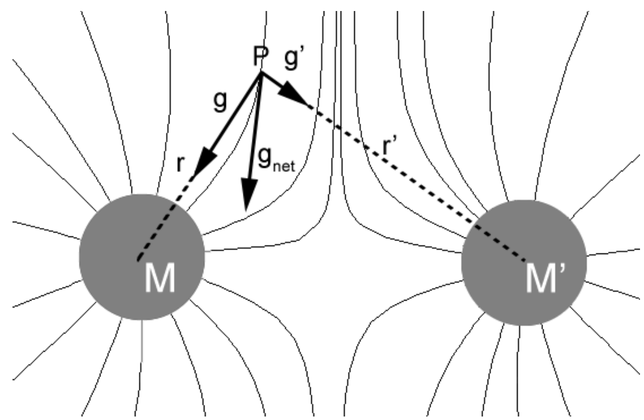


Gravitational fields, p. 5

Consider the condition that the mass, *m*, be small enough that it has a negligible effect on the field of the source object. We can easily imagine a situation where *m* is comparable to *M*. In this case, the field lines are no longer radially symmetric. In the extreme case that *m* = *M*, the field would look like the one to the right. In this case, we would could no longer say that the field is due to *M* alone, since both masses make significant contributions.



Let's take a closer look at this situation. See the diagram to the right. We'll call the masses *M* and *M'*. Suppose we want to determine the field of the masses at a point P, which is distance, *r*, from *M* and distance, *r'*, from *M'*. This is a simple matter of vector addition. The net field, $\mathbf{g}_{net}$, is the vector sum of $\mathbf{g}$ and $\mathbf{g'}$.

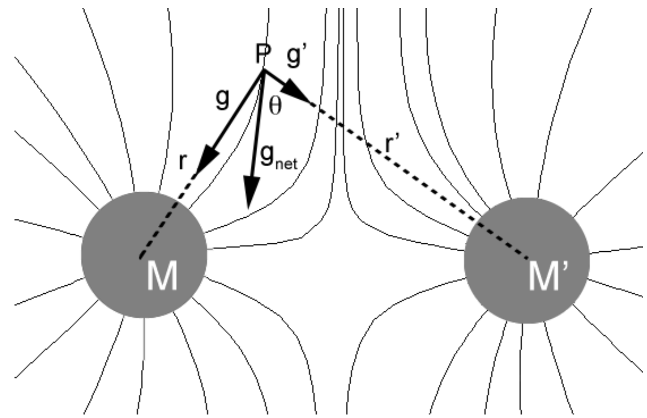


Gravitational fields, p. 6

Note that the direction of $\boldsymbol{g}_{net}$ is tangent to the field line that passes through point P. This is also the direction of the net gravitational force on a small mass, *m*, placed at point P. The magnitude of the net force is

$$F_{net,g} = mg_{net}.$$

This method of adding the fields of individual masses to obtain the field of the mass distribution is called *superposition of fields*.



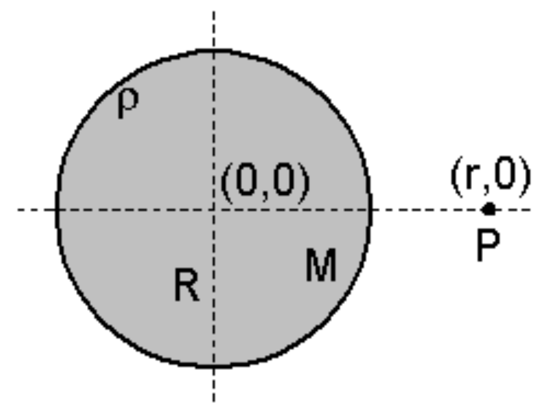
Gravitational fields, p. 7

In what we've done up to this point, we've treated the spherical masses as point masses for the sake of calculating gravitational fields and forces. This is not one of those assumptions made for the sake of convenience. It can be proven (calculus is required) that for points outside of a spherical mass distribution, the entire mass acts as if it were concentrated at a point at the center of the distribution.

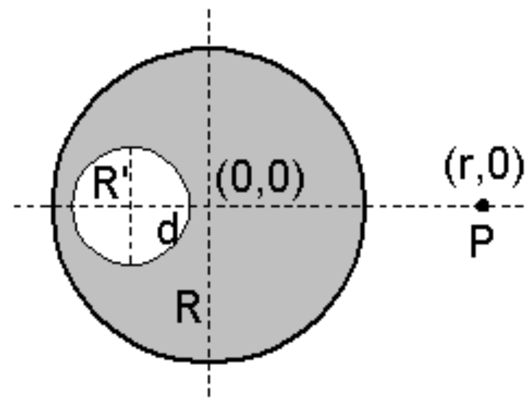
Let's look now at a situation in which we have to consider the mass as an extended body.

Consider the situation shown to the right. A spherical mass of mass, *M,* has radius, *R*, and uniform density, ρ. We know that the gravitational field at point P a distance, *r*, from the center of *M* is given by

$$g(r) = GM/r^2.$$

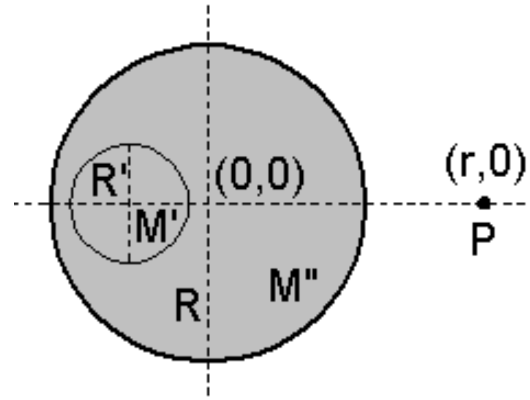


Let's suppose we hollow out a spherical cavity of radius *R'* centered a distance *d* to the left of the center of the larger sphere. What is the gravitational field at point P now? The field will most certainly be less, since there's less mass; however, the distribution of the remaining mass is no longer spherically symmetric. Thus, we can't treat the mass as a point centered at (0,0). We need a different approach.



Gravitational fields, p. 9

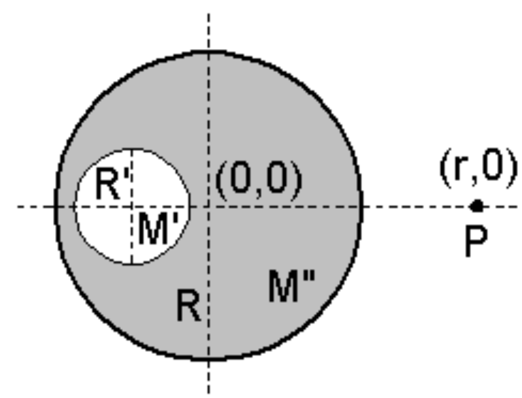
The approach will be to use superposition of fields. Let's first fill the cavity back in with mass but act as if there are two different objects, the filled-in cavity with mass *M'* and the remainder of the mass of the original sphere, $M'' = (M - M')$. We can say that the net field at point P due to the two masses is the following



$$g = g' + g''.$$

Now our goal is to find the field *g"* with the cavity carved out. Thus we solve for *g"*.

$$g'' = g - g'$$

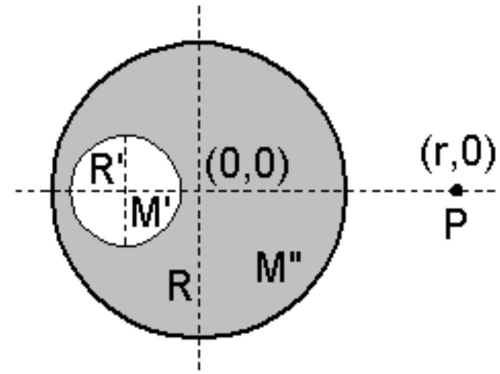


Note that both *g* and *g'* are the fields of spherically-symmetric mass distributions. Thus, we can write expressions for those fields.

$$g = GM/r^2$$

$$g' = GM'/(r + d)^2$$

Gravitational fields, p. 10



Thus, the field at point P with the cavity carved out is

$$g'' = g - g'$$
$$= GM/r^2 - GM'/(r + d)^2.$$

The mass M' that was carved out can be expressed in terms of M using a scaling relationship: $M' = M(R'/R)^3$.

The above is a result of the fact that the density is uniform, and the volume is spherical. (Mass scales as the cube of the scale factor when density is constant.) With this substitution, we obtain

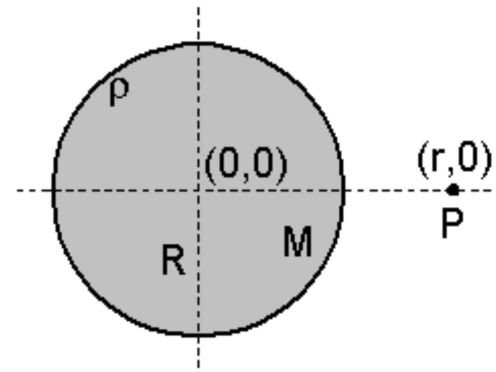
$$g'' = GM[1/r^2 - (R'/R)^3/(r + d)^2].$$

While the result may seem complicated, the relationships used to obtain it are simple.

- The gravitational field of a spherical mass distribution is $g = GM/r^2$.
- The net field of a distribution of masses is a simple superposition.
- Mass is proportional to volume for constant density.
- The volume of a sphere is proportional to the cube of the radius.

Gravitational fields, p. 11

For reference: $g'' = GM[1/r^2 - (R'/R)^3/(r + d)^2]$

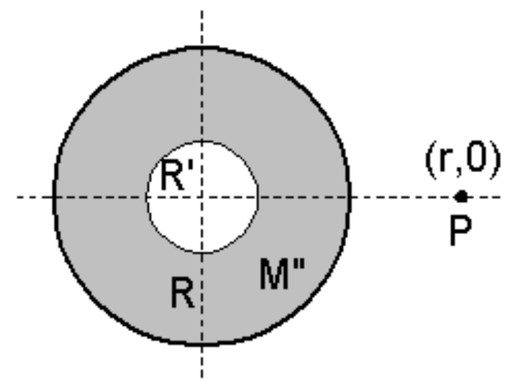


As always, we need to test the result for special cases.
If $R' = 0$, there is no cavity and

$$g'' = GM(1/r^2 – 0) = GM/r^2 = g$$

As expected, the field is the same as for the original sphere.

If $d = 0$, the cavity is centered on the original sphere as shown to the right. The formula for g'' reduces to the following:



$$g'' = GM[1/r^2 - (R'/R)^3/r^2]$$

$$= (GM/r^2)[1 - (R'/R)^3]$$

$$= g[1 - (R'/R)^3]$$

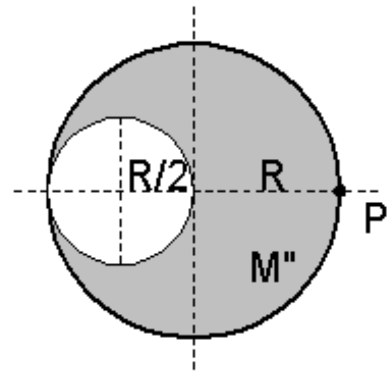
With $R' = 0$, we have $g'' = g$ as before.
With $R' = R$, we have $g'' = 0$. This makes sense, because there's no mass left to produce a field.
With $R' = R/2$, we have $g'' = (7/8)g$. This makes sense, because we're carving out one-eighth of the original mass.

Gravitational fields, p. 12

For reference: $g'' = GM[1/r^2 - (R'/R)^3/(r + d)^2]$

Now let's solve for g'' using these given values:

$$d = R' = R/2 \text{ and } r = R$$



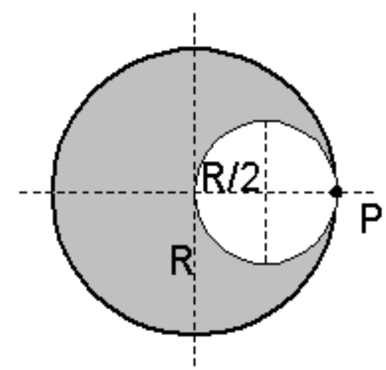
The situation is shown to the right. Substituting the given information into the formula, we have:

$$g'' = GM\{1/R^2 – [(R/2)^3/R^3]/(R + R/2)^2\}$$

$$= GM[1/R^2 – (1/2)^3/(3R/2)^2]$$

$$= (GM/R^2)[1 – (1/8)/(9/4)]$$

$$= (17/18)g$$



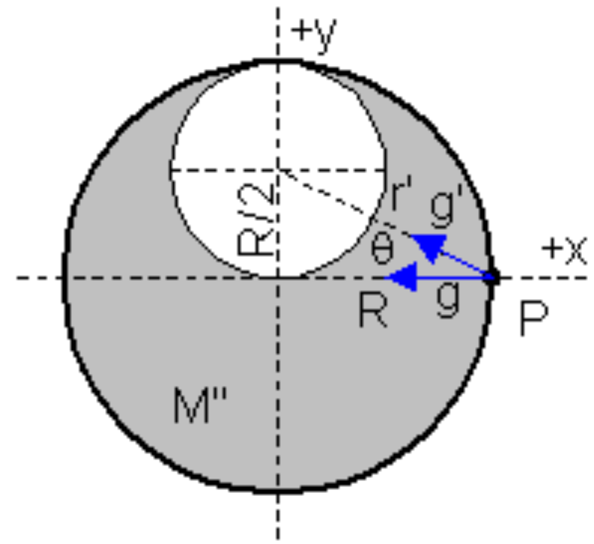
Suppose the cavity had been located as shown to the right. Then all that would change would be one sign shown in red below.

$$g'' = GM\{1/R^2 – [(R/2)^3/R^3]/(R - R/2)^2\}$$

In this case, g'' would simplify to $g/2$. It makes sense that the field would decrease significantly if there were much less mass directly below your feet.

Problem A.

Suppose now that the cavity were positioned as shown to the right. The gravitational field at point P due to the scooped out mass, *M’*, would be directed at an upward angle, while the gravitational field due to *M* would remain the same as before.



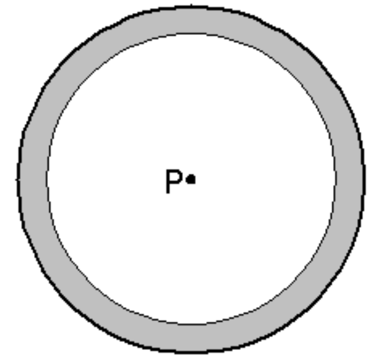
As in the foregoing discussion, *g”* represents the gravitational field of the modified sphere, that is, the object of mass *M”* = *M* – *M’* (original sphere of mass *M* after mass *M*’ is scooped out).

a. Determine the component of ***g”*** along the x-axis in terms of *g* and a numerical factor.

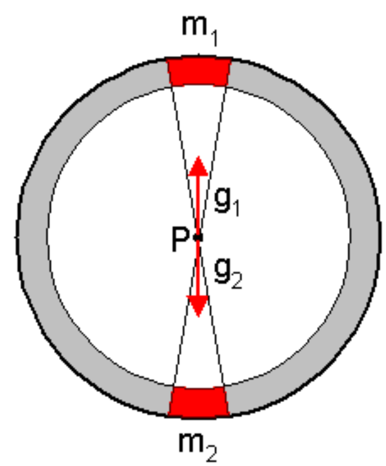
b. Determine the component of ***g”*** along the y-axis in terms of *g* and a numerical factor.

c. Determine the angle that ***g”*** makes with the +x axis. (Note that this is not the angle θ shown in the diagram.)

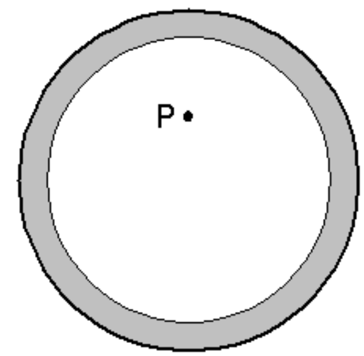
Everything that preceded this had to do with the field outside of the source object. Now we consider the situation inside the source. We first scoop out a very large cavity centered on the object, leaving just a ring of mass as shown to the right. What is the field at point P? That is, would what the force on a small mass placed at the center of ring be?



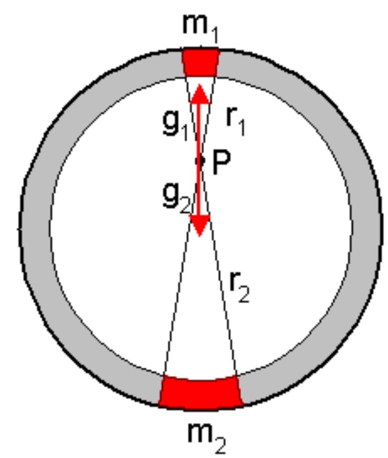
The answer is 0. Let's see why. In the bottom diagram, note the two lines drawn through point P and intersecting the ring at top and bottom. Think of these lines as defining the boundaries of cones (in 3 dimensions) with vertices at P. If the vertex angle of the cone is small, the sections of the ring intersected by the cone can be considered as point masses, m_1 and m_2. The fields $\mathbf{g_1}$ and $\mathbf{g_2}$ due to these masses are equal in magnitude because $m_1 = m_2$, and the distances to point P are the same.



Thus, the net field at point P due to m_1 and m_2 is 0. The same argument could be made no matter what pair of lines were drawn through point P. This means that for every m_1, there is a corresponding m_2 that cancels m_1's field. Hence, the net field at point P is 0.



While the preceding argument may seem obvious, it's not so obvious that a similar argument can be made for any point in the hollowed-out cavity. The net gravitational field at point P in the figure to the right is also 0.

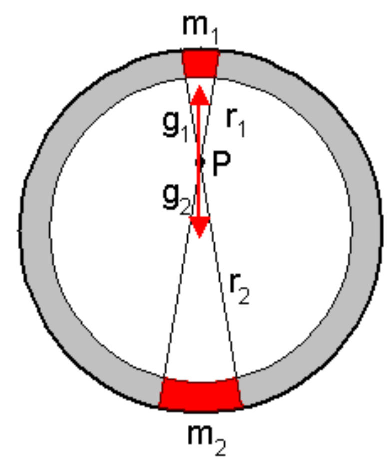


We start the argument in the same way as before by drawing lines through point P and extending them to intersect the ring. See the lower diagram. Since P is off-center, $m_1 < m_2$. Making the approximation before that, m_1 and m_2 are point masses (remember, we can shrink the vertex angle of the cone as small as we want), the corresponding fields at point P are the following:

$$g_1 = Gm_1/r_1^2 \text{ and } g_2 = Gm_2/r_2^2.$$

In 3 dimensions, m_1 and m_2 represent circular patches of mass. The diameter of a patch is proportional to the distance to point P. Since area scales as the square of the scale factor (r in this case), the area of a patch scales as r^2. Since the thickness of the shell is the same for all patches, then the mass of a patch scales as r^2 also.

Gravitational fields, p. 16



This means that we can say $m_1/m_2 = r_1^2/r_2^2$ or $m_1 = m_2r_1^2/r_2^2$. Let's make this substitution into $g_1 = Gm_1/r_1^2$.

$$\begin{aligned} g_1 &= Gm_1/r_1^2 \\ &= G(m_2r_1^2/r_2^2)/r_1^2 \\ &= Gm_2/r_2^2 \\ &= g_2 \end{aligned}$$

Thus, we obtain the same result as when P was at the center of the ring. It doesn't matter where P is as long as it's in the hollowed-out region. The net gravitational field is still 0.

Note that this result is a consequence of canceling dependencies on r^2. One of the dependencies is geometrical in nature, that of the mass depending on the square of r. The other dependency is that of gravitational force depending on $1/r^2$.

Let's be clear about what we've shown. We've shown that the net gravitational field due to a uniform spherical ring of matter is 0 at any point within the ring. This only applies to the ring. Any object with mass beyond the ring will contribute to a field at point P. That's because it's impossible to shield objects from gravitational influences.

Gravitational fields, p. 17

Part of the power of physics is being able to use fairly simple relationships to reach general conclusions and to extend those conclusions to similar situations. An argument similar to the one just made for gravitational fields also applies to electrical fields. That's because electrical force also depends on $1/r^2$. The forces that charged particles exert on each other decrease with the inverse square of the separation of the particles. Thus, the electric field inside a ring of charge will be 0.

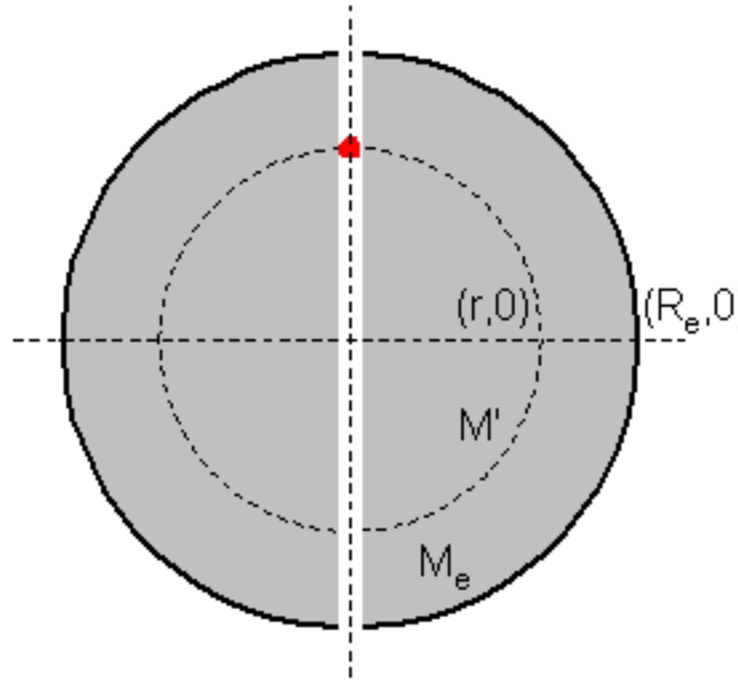
In the case of electric fields, the 0 is absolute in the sense that electric fields due to other charged sources outside the ring can be shielded. The reason this is possible is that electric charge comes in two varieties, positive and negative. Forces between like charges are repulsive and between opposite charges are attractive. Thus, the electric forces and fields due to charges can cancel as a result of being opposites. This could never happen for gravitational force. There's only one kind of mass, and the gravitational force between masses is always attractive.

Simple Harmonic Motion

The next situation we consider involves objects inside of other objects. Let's suppose we were able to drill a tunnel along the axis of the Earth as shown to the right. We'll assume that the mass removed is insignificant and can be ignored compared to the total mass, M_e, of the Earth.

Now we drop a small ball of mass, *m*, into the tunnel. What motion will the ball have? Will it slow to a stop at the center of the Earth? Will it keep going and turn around at some point? Will it keep going and never return?

In order to determine the motion, we must first know what force the ball experiences. Thus, we need to know what the gravitational field is when the ball is at distance, *r*, from the center of the Earth.

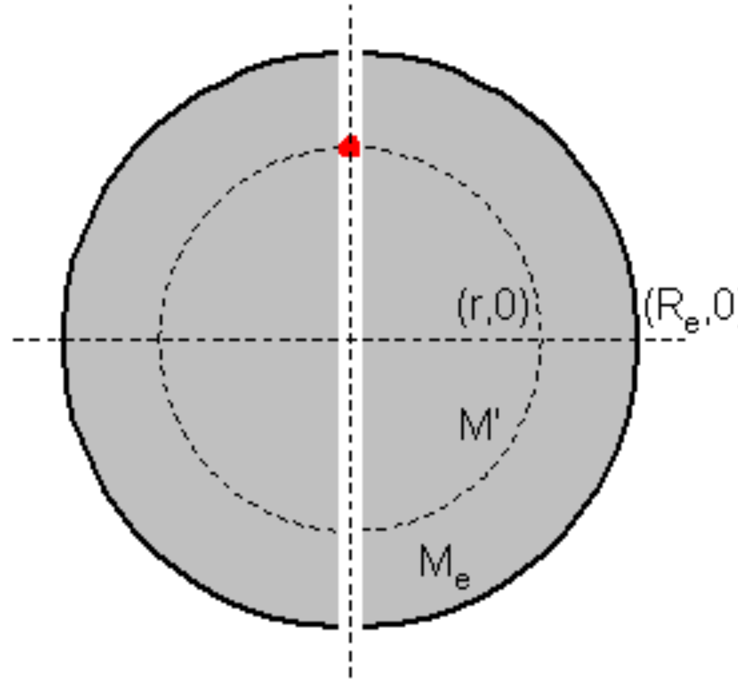


SHM, p. 2

In order to determine the field, we use this postulate:

> The gravitational field at a distance *r* from the center of a sphere of mass *M* of radius *R* and uniform density is due only to the mass enclosed by a sphere of radius *r*.

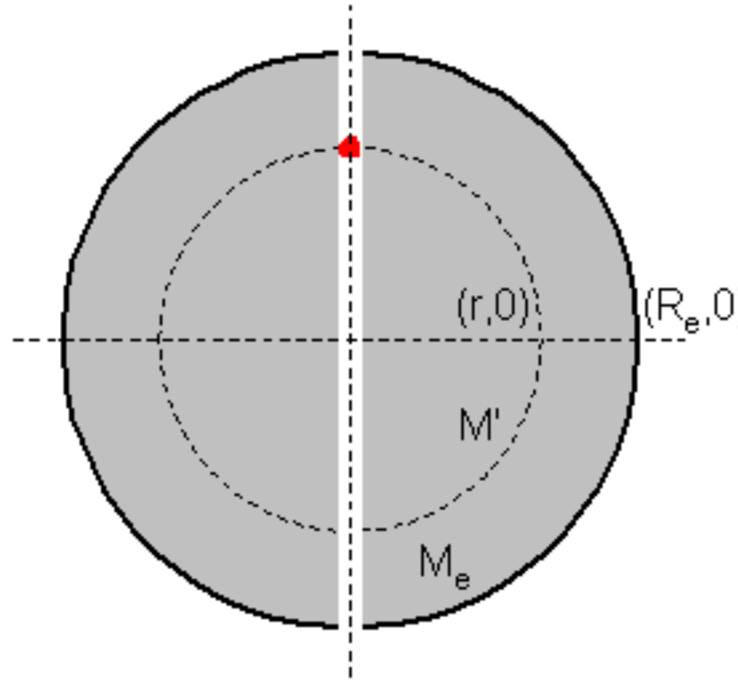
The above should make sense based on what we showed earlier about how the ring of matter with $r > R$ contributes a net field of 0. Now let's see what this means for how the gravitational field depends on *r*.



SHM, p. 3

The gravitational field due to the mass, M', enclosed by a sphere of radius, r, is the following:

$$g' = GM'/r^2.$$



Note that we no longer take the mass as a constant, since M' depends on r. Using a similar scaling argument as before,

$$M' = M_e(r^3/R_e^3)$$

Substituting, we obtain

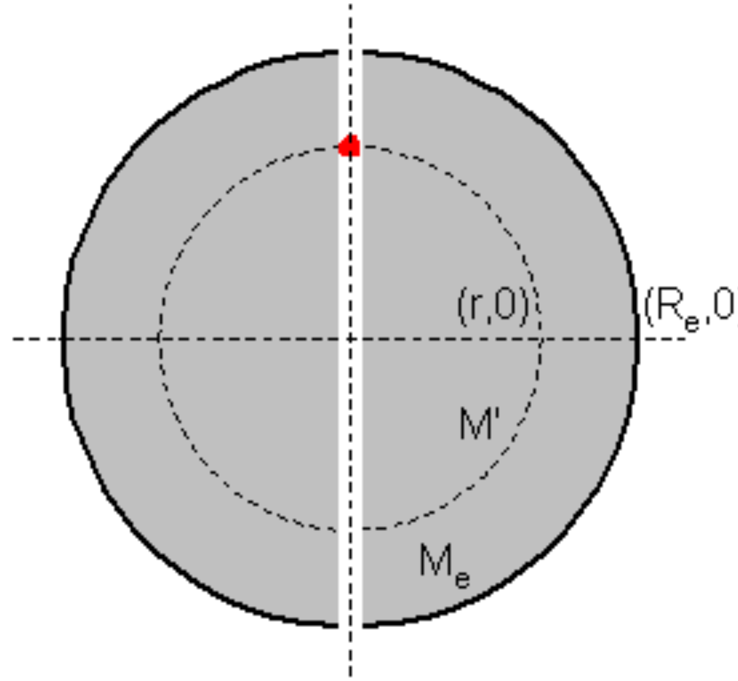
$$g' = GM_e(r^3/R_e^3)/r^2$$

$$= (GM_e/R_e^3)r$$

This says that g depends linearly on r, since the collection of factors preceding the r is a constant. Note that we can re-express this collection of constants as $GM_e/R_e^3 = g/R_e$, where g is the field at the surface of the Earth. Thus, we have the following result for g':

$$g' = (g/R_e)r$$

We're now in a position not only to describe the motion of the ball in words but also to write an equation for its position as a function of time. This is another example of the power of physics. We've actually already solved this problem. Here's how we know.



Remember that we showed $g' = (g/R_e)r$. We can also write this as

$$g' = Cr, \text{ where } C = g/R_e.$$

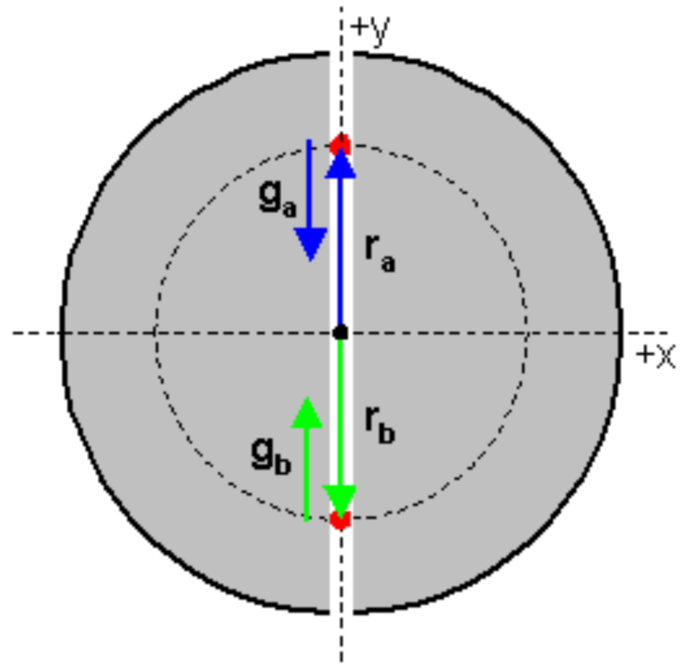
Now recall the conditions for simple harmonic motion. There are two of them.

1. The force tending to restore an object to its equilibrium position is directly proportional to the displacement of the object from the equilibrium position.
2. The force is a restoring force in the sense that the force and displacement are always in opposite directions.

Let's see how this applies to the ball in the Earth tunnel.

To reiterate, here are the conditions for SHM.

1. The force tending to restore an object to its equilibrium position is directly proportional to the displacement of the object from the equilibrium position.
2. The force is a restoring force in the sense that the force and displacement are always in opposite directions.



Here's why the conditions apply to the ball in the Earth tunnel.

1. The equilibrium position is the center of the Earth. This is where the force on the ball is 0. Thus, *r* represents the displacement of the ball from equilibrium. Since $g' = Cr$ and C is a constant, the first condition is satisfied.
2. For +y (see diagram above), the displacement from equilibrium, $\boldsymbol{r_a}$, is positive while the field, $\boldsymbol{g_a}$, is negative. For –y, the displacement from equilibrium, $\boldsymbol{r_b}$, is negative while the field, $\boldsymbol{g_b}$, is positive. Thus, the second condition for SHM is satisfied.

Now that we know the motion is simple harmonic, we can immediately write expressions for the period of the motion and the function $y(t)$.

For any object in SHM, the period is given by this formula:

$$T = 2\pi[(\text{Inertial Property})/(\text{Elastic Property})]^{0.5}$$

Think of the Inertial Property as something that resists motion and tends to increase the period. For a spring, that's the mass of the object. For a simple pendulum the Inertial Property is the length of the string.

The Elastic Property is something that tends to make the motion "springier" and decrease the period. For a spring, that's the spring constant. For a pendulum, it's the value of the gravitational field.

Spring: $T = 2\pi(m/k)^{0.5}$ Pendulum: $T = 2\pi(L/g)^{0.5}$

The reason we can say this with confidence is that the same mathematics is used to solve all problems involving a Hooke's Law type of restoring force. The only differences come in the constants.

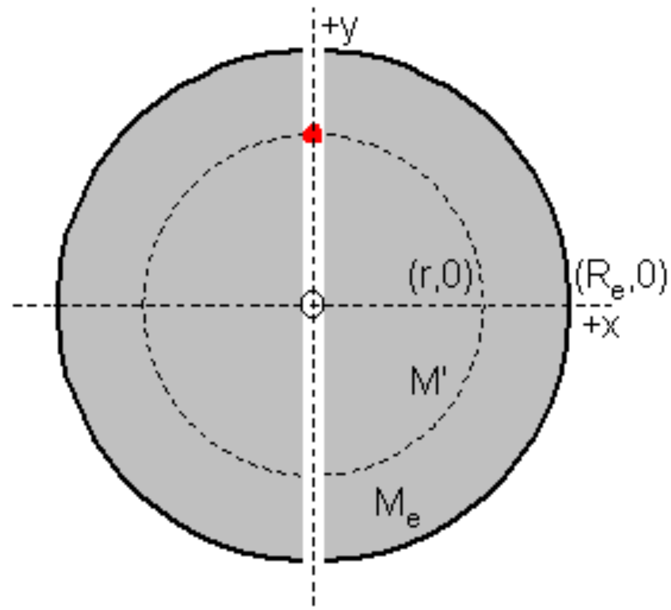
Let's now apply this to the Earth tunnel. We saw that the magnitude of the field at distance *r* is

$$g' = Cr, \text{ where } C = g/R_e.$$

The constant *C* here plays exactly the same role as that in the acceleration of an object of mass, *m*, oscillating on a spring. The magnitude of that acceleration is

$$a = |(k/m)x|,$$

where *x* is the displacement from equilibrium.



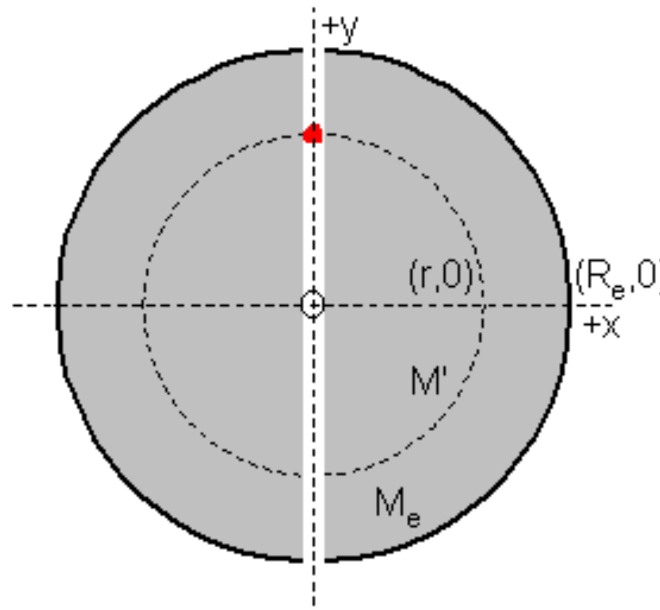
The constant, *C*, like *k/m*, is the proportionality constant between acceleration and displacement. Thus, we can write for the period of the ball in the Earth tunnel:

$$T = 2\pi(1/C)^{0.5}$$

$$= 2\pi(R_e/g)^{0.5}$$

Note that like the simple pendulum, a distance, R_e in this case, plays the role of the Inertial Property, tending to make the period greater, while *g* plays the role of the Elastic Property.

Next we write the equation of motion of the ball. We know that the form of the equation is the same as that for SHM in general:



$$y(t) = A\cos(\omega t + \phi) + y_{eq}$$

For the Earth tunnel, we have $A = R_e$ for the amplitude of the motion, and $y_{eq} = 0$ for the equilibrium position. We also know that

$$\omega = 2\pi/T = 2\pi/[2\pi(R_e/g)^{0.5}]$$
$$= (g/R_e)^{0.5}$$

The phase is determined by noting that at $t = 0$, the ball is at position $y = +R_e$. Substituting,

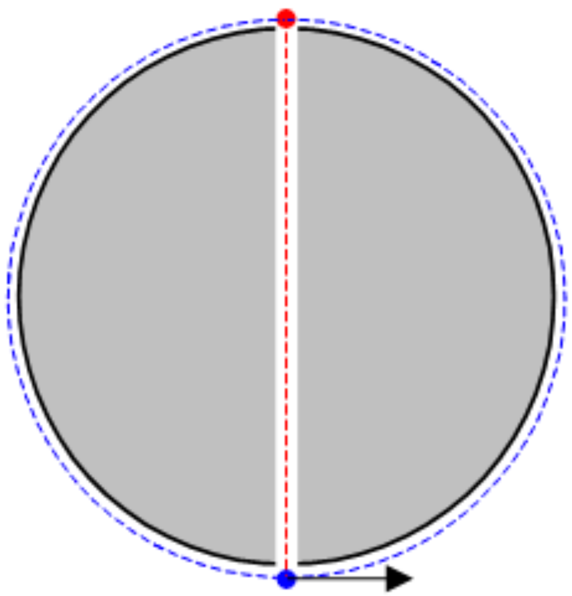
$$y(0) = A\cos(\phi)$$

The phase, ϕ, must therefore be 0. (Note that if we had used a sine function to describe $y(t)$, then the phase would have to be $\pi/2$, since $\sin(\pi/2) = 1$.)

Putting it all together, the equation of motion of the ball in the Earth tunnel is:

$$y(t) = R_e\cos[(g/R_e)^{0.5}t]$$

Problem B.



The diagram shows 2 small spheres, each of mass, *m*, one red and one blue. The larger gray sphere is a planet of mass *M* and radius *R*. A tunnel is drilled through the planet along its rotation axis. The planet is unusual in being perfectly smooth and spherical with no atmosphere.

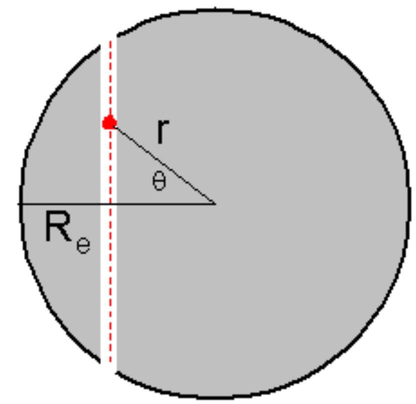
The red sphere is dropped at the upper opening of the tunnel at the same instant that that the blue sphere passes the lower end of the tunnel in a circular orbit around the planet.

Assume that the diameter of the blue sphere is negligible so that we can say that the radius of its orbit is *R*.

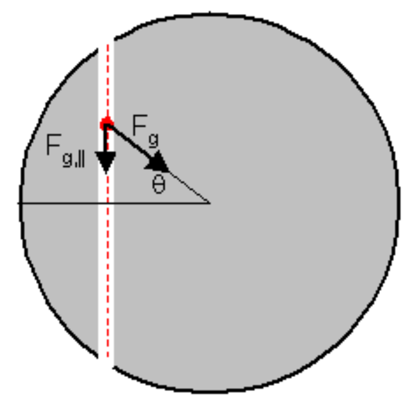
a. When the red sphere has reached the bottom end of the tunnel, where is the blue sphere in its orbit? In order to answer, you'll need to determine expressions for the period of motion of each sphere and compare results.
b. Determine expressions in terms of *g* and *R* for the speed of the blue sphere and the maximum speed of the red sphere. Compare results.

Problem C.

a. Prove that the period of oscillation of a small object dropped into an Earth tunnel drilled off axis is the same as that in a tunnel coincident with the Earth's axis. Ignore any frictional forces between the object and the sides of the tunnel.

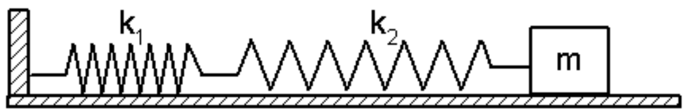


As an aid to solving the problem, consider the lower diagram, which shows the gravitational force, $\boldsymbol{F_g}$, acting on the object. The restoring force is the component of $\boldsymbol{F_g}$ parallel to the tunnel.

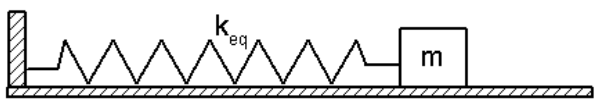


b. Determine the numerical value of the period of oscillation of the object.

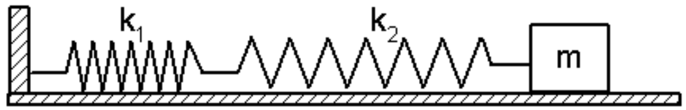
We'll do one more problem to illustrate the SHM method. The situation is illustrated below. Two springs of different spring constants k_1 and k_2 are connected together in a series arrangement and then attached to a block of mass, m. The system is set into horizontal oscillation on a frictionless table. Determine the period of the motion in terms of k_1, k_2, and m.



Solving the problem amounts to finding an equation for the equivalent spring constant, k_{eq}. That is, if the two springs were represented by a single spring as shown below, what would its spring constant have to be so that the mass had exactly the same motion as for the two springs in series.



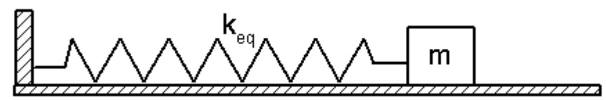
Finding the equivalent spring constant requires us to examine the forces. First, both of the springs 1 and 2 produce the same restoring force. We know this, because the tension is the same in all parts of the spring. However, springs 1 and 2 will, in general, stretch by different amounts under the action of the same force.



Let's say that under the application of force, F_T, spring 1 stretches by an amount x_1 and spring 2 stretches by an amount x_2. Using Hooke's Law, we can express these amounts in terms of the tension force of the spring constants:

$$x_1 = F_T/k_1 \text{ and } x_2 = F_T/k_2.$$

Now consider the equivalent spring. Under the action of the same force, F_T, the stretch would be the sum of x_1 and x_2. Then,



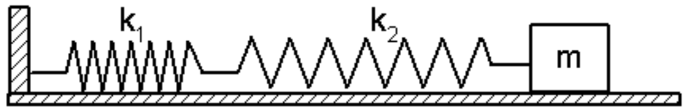
$$x_1 + x_2 = F_T/k_{eq}.$$

We substitute the previous expressions for x_1 and x_2.

$$F_T/k_1 + F_T/k_2 = F_T/k_{eq}$$

Dividing out the force term provides the relationship we're looking for:

$$1/k_{eq} = 1/k_1 + 1/k_2$$



The formula for the period of a mass, *m*, on a spring of spring constant, *k*, is

$$T = 2\pi(m/k)^{0.5}$$

Applied to the series springs, we have $T = 2\pi(m/k_{eq})^{0.5}$ with $1/k_{eq} = 1/k_1 + 1/k_2$.

As a check, suppose $k_1 = k_2 = k$. Then $k_{eq} = k/2$. This means the series of 2 identical springs behaves as a single spring with half the spring constant of either individual spring. Under the action of the same force, F_T, the elongation will be twice as much as for each individual spring. With the equivalent spring being less “springy”, the period will be greater. In this case, the period will be sqrt(2) times as great.

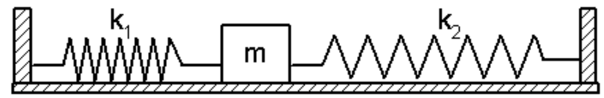
Let’s substitute $k_1 = k$ and $k_2 = 2k$ as another example. Then

$$1/k_{eq} = 1/k + 1/(2k) = 3/(2k).$$

Then, $k_{eq} = (2/3)k$. Solving for the period,

$$T = 2\pi(m/k_{eq})^{0.5} = 2\pi[3m/(2k)]^{0.5}$$

Problem D.



A block of mass, *m*, is connected between two springs of spring constants k_1 and k_2. The block oscillates in simple harmonic motion on a horizontal, frictionless surface.

a. Determine the period of oscillation of the block in terms of *m*, k_1 and k_2.

b. Check your result for the special case of $k_1 = k_2$.

c. Given that $k_1 = k$ *and* $k_2 = 2k$, obtain an expression for the period of oscillation in terms of *m* and *k*.